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Complex Lie Algebroids and ACH manifolds

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Abstract

We propose the definition of a Manifold with a complex Lie structure at infinity. The important class of ACH manifolds enters into this class.

Contents

1 Manifolds with a complex Lie structure at infinity

Manifolds with a Lie structure at infinity are well known in literature [2, 1, 3]. Remind the definition. Let X be a smooth non compact manifold together with a compactification $X \hookrightarrow \overline{X}$ where $\overline{X}$ is a compact manifold with corners[1]. A Lie structure at infinity on X is the datum of a Lie subalgebra $\mathcal{V}$ of the Lie algebra of vector fields on $\overline{X}$ subjected to two restrictions

1. every vector field in $\mathcal{V}$ must be tangent to each boundary hyperface of $\overline{X}$.

2. $\mathcal{V}$ must be a finitely generated $C^\infty(\overline{X})$–module this meaning that exists a fixed number k such that around each point $x \in \overline{X}$ we have for every $V \in \mathcal{V}$,

$$\varphi(V - \sum_{i_1}^{k} \varphi_k V_k) = 0$$

where φ is a function with $\varphi = 1$ in the neighborhood, the vector fields $V_1, ..., V_n$ belong to $\mathcal{V}$ and the coefficients φ_j are smooth functions with univoquely determined germ at x.

By The Serre–Swann equivalence there must be a Lie Algebroid over $\overline{X}$ i.e. a smooth vector bundle $A \longrightarrow \overline{X}$ with a Lie structure on the space of sections $\Gamma(A)$ and a vector bundle map $\rho : A \longrightarrow \overline{X}$ such that the extended map on sections is a morphism of Lie algebras and satisfies

[1] with embedded hyperfaces i.e the definition requires that every boundary hypersurface has a smooth defining function

1. $\rho(\Gamma(A)) = A$
2. $[X, fY] = f[X, Y] + (\rho(X)f)Y$ for all $X, Y \in \Gamma(A)$.

In particular we can define manifolds with a Lie structure at infinity as manifolds X with a Lie algebroid over a compactification $\overline{X}$ with the image of ρ contained in the space of boundary vector fields (these are called boundary Lie algebroids). Notice that the vector bundle A can be "physically reconstructed" in fact its fiber A_x is naturally the quotient $\mathcal{V}/\mathcal{V}_x$ where

$$\mathcal{V}_x := \Big\{ V \in \mathcal{V} : V = \sum_{\text{finite}} \varphi_j V_j,\, V_j \in \mathcal{V},\, \varphi_j \in C^\infty(\overline{X}),\, \varphi_j(x) = 0 \Big\}.$$

In particular since over the interior X there are no restrictions $\rho : A_{|X} \longrightarrow TX$ is an isomorphism. In mostly of the applications this map degenerates over the boundary. One example for all is the Melrose b–geometry [8] where one takes as $\overline{X}$ a manifold with boundary and $\mathcal{V}$ is the space of all vector fields that are tangent to the boundary. Here $A = {}^bT\overline{X}$ the b–tangent bundle. In fact all of these ideas are a formalization of long program of Melrose.

In this section we aim to take into account complex Lie algebroids i.e. complex vector bundles with a structure of a complex Lie algebra on the space of sections and the anchor mapping ($\mathbb{C}$–linear, of coarse) with values on the complexified tangent space $T_{\mathbb{C}}\overline{X} = T\overline{X} \otimes \mathbb{C}$.

Definition 1.1 — A manifold with a complex Lie structure at infinity is a triple $(X, \overline{X}, A)$ where $X \hookrightarrow \overline{X}$ is a compactification with a manifold with corners and $A \longrightarrow \overline{X}$ is a complex Lie algebroid with the $\mathbb{C}$–linear anchor mapping $\rho : A \longrightarrow T_{\mathbb{C}}\overline{X}$ with values on the space of complex vector fields tangent to each boundary hypersurface.

Note that over the interior the algebroid A reduces to the complexified tangent bundle so a hermitian metric along the fibers of A restricts to a hermitian metric on X. We shall call the corresponding object a **hermitian manifold with a complex Lie structure at infinity** or a hermitian Lie manifold.

2 ACH manifolds

The achronim ACH stands for asymptotically complex hyperbolic manifold. This is an important class of non–compact Riemannian manifolds and are strictly related to some solutions of the Einstein equation [6, 4] and CR geometry [5]. We are going to remind the definition. Let $\overline{X}$ be a compact manifold of even dimension $m = 2n$ with boundary Y . We will denote by X the interior of X, and choose a defining function u of Y , that is a function on $\overline{X}$, positive on X and vanishing to first order on $Y = \partial\overline{X}$. The notion of ACH metric on X is related to the data of a strictly pseudoconvex CR structure on Y, that is an almost complex structure J on a contact distribution of Y, such that $\gamma(\cdot, \cdot) = d\eta(\cdot, J\cdot)$ is a positive Hermitian metric on the contact distribution (here we have chosen a contact form η). Identify a collar neighborhood of Y in X with $[0, T) \times Y$, with coordinate u on the first factor. A Riemannian metric g is defined to be an ACH metric on X if there exists a CR structure J on Y, such that near Y

$$g \sim \frac{du^2 + \eta^2}{u^2} + \frac{\gamma}{u}. \tag{1}$$

The asymtotic $\sim$ should be intended in the sense that the difference between g and the model metric $g_0 = \dfrac{du^2 + \eta^2}{u^2} + \dfrac{\gamma}{u}$ is a symmetric 2–tensor κ with $|\kappa| = O(u^{\delta/2})$, $0 < \delta \leq 1$. One also

requires that each g_0–covariant derivative of κ must satisfy $|\nabla^m \kappa| = O(u^\delta/2)$. The complex structure on the Levi distribution H on the boundary is called **the conformal infinity** of g. Hereafter we shall take the normalization

$$\delta = 1.$$

This choice is motivated by applications to the ACH Einstein manifolds where well known normalization results show its naturality [6].

2.1 The square root of a manifold with boundary

In order to show that ACH manifolds are complex Lie manifolds we need a construction of Melrose, Epstein and Mendoza [7]. So let $\overline{X}$ be a manifold with boundary with boundary defining function u. Let us extend the ring of smooth functions $C^\infty(\overline{X})$ by adjoining the function $\sqrt{u}$. Denote this new ring $C^\infty(\overline{X}_{1/2})$ In local coordinates a function is in this new structure if it can be expressed as a C^∞ function of $u^{1/2}, y_1, ..., y_n$ i.e. it is C^∞ in the interior and has an expansion at $\partial\overline{X}$ of the form

$$f(u,x) \sim \sum_{j=0}^{\infty} u^{j/2} a_j(x)$$

with coefficients $a_j(x)$ smooth in the usual sense. The difference $f - \sum_{j=0}^{N} u^{j/2} a_j(x)$ becomes increasingly smooth with N. In this way f is determined by the asymtotic series up to a function with all the derivatives that vanish at the boundary. Since the ring is independent from the choice of the defining function and invariant under diffeomorphisms of $\overline{X}$ the manifold $\overline{X}$ equipped with $C^\infty(\overline{X}_{1/2})$ is a manifold with boundary globally diffeomorphic to $\overline{X}$.

Definition 2.2 — The square root of $\overline{X}$ is the manifold $\overline{X}$ equipped with the ring of functions $C^\infty(\overline{X}_{1/2})$. We denote it $\overline{X}_{1/2}$

Notice the natural mapping $\iota_{1/2} : \overline{X} \longrightarrow \overline{X}_{1/2}$ descending from the inclusion $C^\infty(\overline{X}) \hookrightarrow C^\infty(\overline{X}_{1/2})$ is not a C^∞ isomorphism since it cannot be smoothly inverted. Note also the important fact that the interiors and boundaries of $\overline{X}$ and $\overline{X}_{1/2}$ are canonically diffeomorphic. The change is the way the boundary is attached.

2.2 The natural complex Lie algebroid associated to an ACH manifold

Let X be an orientable $2n$–dimensional ACH manifold with compactification $\overline{X}$, define $Y := \partial\overline{X}$ and remember for further use it is canonically diffeomorphic to the boundary of $\overline{X}_{1/2}$. So Y is a CR $(2n-1)$– manifold with contact form η (we keep all the notations above). Let $H = \operatorname{Ker}\eta$ the Levi distribution with choosen complex structure $J : H \longrightarrow H$. Extend J to a complex linear endomorphism $J : T_{\mathbb{C}}Y \longrightarrow T_{\mathbb{C}}Y$ with $J^2 = -1$. Define the complex subundle $T_{1,0}$ of $T_{\mathbb{C}}Y$ as the bundle of the i–eigenvectors. Notice that directly from the definition on the CR structure it is closed under the complex bracket of vector fields; for this reason the complex vector space

$$\mathcal{V}_{1,0} := \{V \in \Gamma(\overline{X}_{1/2}, T\overline{X}_{1/2}) : V_{|Y} \in \Gamma(T_{1,0})\}$$

is a complex Lie algebra. It is also a finitely generated projective module. To see this, around a point $x \in Y$ let $U_1, ..., U_r$, $r = 2(n-1)$ span H and let $T \in \Gamma(Y, TY)$ be the Reeb vector

field, univoquely determined by the conditions $\gamma(T) = 1$ and $d\gamma(\cdot, T) = 0$. Then it is easy to see that the following is a local basis of $\mathcal{V}_{1,0}$ over $C^\infty(\overline{X}_{1/2}, \mathbb{C})$:

$$\sqrt{u}\partial_u,\ U_1 - iJU_1,\ ...,\ U_r - iJU_r,\ \sqrt{u}T \tag{2}$$

where u is a boundary defining function. Now let

$$\widetilde{\mathcal{V}}_{\mathrm{ACH}} := \sqrt{u}\mathcal{V}_{1,0}$$

the submodule defined by the multiplication of every vector field by the smooth function $\sqrt{u}$. A local basis corresponding to (2) is

$$u\partial_u,\ \sqrt{u}[U_1 - iJU_1],\ ...,\ \sqrt{u}[U_r - iJU_r],\ uT. \tag{3}$$

Let $A \longrightarrow \overline{X}_{1/2}$ the corresponding Lie algebroid. The following result is immediate

Theorem 2.2 — Every ACH metric on X extends to a smooth hermitian metric on A. In particular an ACH manifold is a manifold with a Complex Lie structure at infinity.

Proof — Just write the matrix of the difference κ on a frame of the form (3). This gives the right asymptotic. □

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